Division of a Contest with Identical Prizes*

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Abstract

This paper studies an economic contest with identical prizes. We consider the effects of division of the contest. When the contest designer divides the contest symmetrically, each participant competes in each assigned division. The main result is that division is sometimes profitable for the contest designer, in the sense that division brings about increases in lower ability agents' (and occasionally higher ability agents') efforts in exchange for decreases in other agents' efforts. We also study an application to educational attainment. The result helps to explain why it is difficult to find class size effects in empirical data.

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1 Introduction

In a large literature, economic contests have been studied under many situations. Well known applications are job promotion in internal labor markets, rent seeking activities, and R&D rivalry.[1] Many studies assume the contest structure to be given and study the equilibrium strategies of participants. However, contest design problems from the viewpoint of the contest designer have attracted much attention recently. Gradstein and Konrad (1999) compared multistage contests with simultaneous contests. Clark and Riis (1998) studied complete information contests with identical prizes and compared simultaneous distributions of prizes with sequential distributions of prizes. Moldovanu and Sela (2001) studied the optimal allocation of prizes and provided the condition for a contest with multiple positive prizes to be optimal for the contest designer.[2]

The aim of this paper is to study the effects of division of a contest.[3] An example of division is the following. Consider a contest with four participants and two identical prizes. If the contest designer divides the contest, each participant competes with one of his fellow participants for one prize. If he does not divide the contest, each participant competes with his three fellow participants for two prizes. Thus, the contest designer can determine whether each participant competes in the whole contest or in each assigned division.

In the real world, we can observe many phenomena that can be regarded as division of a contest with identical prizes. Examples include the following:

- A large company has several departments and sections. Peers compete for promotion in each division.
- In the realm of politics, there exist various election systems, including the small electoral district system, the medium-sized system, and the large-sized system. Here, we can regard such elections as divided contests.
- In many schools, grading is conducted in each classroom with grading on a curve.[4]

Now, our questions are "what does division bring about?" and "when and why does the designer of a contest divide the contest?"

[1] An initial study on the economic analysis of contests has been conducted by Lazear and Rosen (1981).

[2] Further literature reviews are provided in Section 1.1.

[3] An independent work by Moldovanu and Sela (2002) considered closely related topics to ours. The contest analyzed in their paper is detailed in Section 1.1.

[4] Classroom education with relative evaluations has contest aspects. Students study to get the higher grade, say, "excellent." In public elementary schools in Japan, grading had been conducted in each classroom with grading on a curve until the school year 2001.

To answer these questions, we study the following contest game. The principal, the contest designer, hires n risk neutral agents for one period and assigns a task for each of them. There exist $k(< n)$ identical prizes. Each agent outlays his effort to get a prize. There are k winners, each of whom obtains one prize. We assume that the numbers n and k are exogenously given. Different agents have different abilities, which affect the marginal costs of their efforts. Each agent's ability is his private information. Abilities are independently and identically distributed. The distribution of abilities is common knowledge for the principal and the agents. The effort level of each agent is observable by all players at the end when the agents have already chosen their effort level.

The principal can divide this contest symmetrically into some contests.[5] More precisely, the principal can divide agents into $a(> 1)$ groups randomly, and then each agent competes in each assigned division. Here $a(\in A)$ is a common divisor of n and k, and A is the set of common divisors. In each division, which consists of n/a participants and k/a prizes, each agent whose effort level is no lower than the k/a-th place gets a prize. The principal chooses the optimal contest structure $a^* \in A$ according to his objective.

Our results are as follows: First, we identify the symmetric Bayesian Nash equilibrium strategy of the contest game. The equilibrium effort levels are strictly increasing with respect to abilities. Note that the contest game considered here is isomorphic to an all-pay auction. Consequently, we can use several existing results in the literature on auction theory to study our contest game.

Second, we demonstrate that division brings about increases in lower ability agents' (and occasionally higher ability agents') efforts in exchange for decreases in other agents' efforts. The fundamental reason why division makes differences in the equilibrium behavior of each type is that division makes room for an inefficient allocation of prizes. An inefficient allocation means that the winners are not chosen in the order of the agents' abilities. If the principal divides a contest, it is possible that the effort level of a winner in one division is smaller than that of a loser in another division. Due to the fact that the agents use the same symmetric equilibrium strategy in every division, the above phenomenon implies an inefficient allocation of prizes. Thus division changes each type's probability of winning, while it does not change the ratio of the number of participants to the number of prizes, and consequently, the equilibrium behavior of each type is changed by division.

Third, we show that division decreases the expected effort per agent under a certain regularity condition on the distribution of abilities (the definition of regularity is described in Section 4.2). This is a direct consequence of the result of the optimal multi-object auction studied by Maskin and Riley (1989).

[5] We assume that n and k are not mutually prime.

Fourth, we show that division is sometimes profitable for the contest designer. This is the main result of the present paper. We analyze the optimal contest structure under the following objective of the principal. The principal maximizes his profit and the profit is the sum of each agent's output. Here each agent's output may not be linear in his effort. We show that, under the regularity condition, division is never profitable for the contest designer if the function that transforms each agent's effort to output is linear. However, division is sometimes profitable for the contest designer if the function is concave or convex, even if the regularity condition holds.

The result in the linear case is straightforward from the third result. Since each agent's output is linear in his effort, the maximization of total output is equivalent to the maximization of total effort. Therefore, we can apply the results of optimal auction design to the study of optimal contest design. Specifically, under the regularity condition, an inefficient allocation of prizes (i.e., dividing) is not profitable. However, in non-linear cases, the maximization of total output is no longer equivalent to the maximization of total effort. Therefore, we cannot apply the results of optimal auction to the optimal contest design under a non-linear setting. Furthermore, if the objective of the principal is not additively separable in each agent's output, the profit maximization for the principal is not achieved by the maximization of total effort. Thus, generally, the optimal contest design is different from the optimal auction design, so that the division of a contest can be profitable even if the regularity condition holds.

Our results have an implication to the study of educational attainment. The relation between educational attainment and class size is studied in a large literature. It has been expected that class size reductions yield an increase in educational productivity. This positive effect is called the class size effect. However, many empirical studies[6] conclude that there exists no significant class size effect.

Our discussion helps to explain why it is difficult to find class size effect in some countries. Suppose that the students are graded on a curve and grading is conducted in each classroom as in Japanese public elementary schools (until the school year 2001). Under such a grading method, we conclude that class size reductions yield a decrease in the expected effort per student. This topic is detailed in Section 6.

The rest of this paper is organized as follows: In the next subsection, we describe related literature. In Section 2, we present the model. In Section 3, we identify the symmetric equilibrium strategy. In Section 4, we study the effects of division. In Section 5, the principal's contest design problem is studied. In Section 6, an application to educational attainment is described. Section 7 concludes.

[6]See the references in Hoxby (2000) and Lazear (2001).

1.1 Related Literature

The contest game considered in this paper can be regarded as a variation of the all-pay auction with incomplete information. Recently, Amann and Leininger (1996) and Krishna and Morgan (1997) studied the all-pay auction and the war of attrition. The former analyzed an asymmetric all-pay auction and the latter analyzed an all-pay auction with affiliated types.

However, the contest game considered in this paper and the all-pay auction have some differences. In the all-pay auction, each bidder's bid is equivalent to his payment as well as the seller's revenue. Therefore, the revenue maximization for the seller is achieved by the maximization of the sum of bidders' bids. However, in situations of contests, the relation between each agent's effort and the principal's profit may not be linear. As a consequence, the principal's optimization problem is probably not the maximization of total effort, and then it is more complicated than that in the all-pay auction.[7]

Singh and Wittman (2001) studied the contest in which each agent's output is non-increasing returns to his effort. Additionally, it is assumed that the participant with the highest output need not win. They described the properties of optimal contests and show that, for an open interval of types, an optimal contest uses the rule that the agent with the highest output wins.

The present paper and Moldovanu and Sela (2001, 2002) considered not only non-increasing returns cases but also increasing returns cases. Moldovanu and Sela (2002) considered closely related topics to ours. The contest analyzed in their paper is the following. The contest designer has a given amount of money for prize. He can split the money to several equal prizes, as well as he can split the participants among several sub-contests. The authors compared the performance of such split schemes to that of grand winner-take-all fashion.

As in ours, the form of the function that transforms each agent's effort to output plays the central role in their analysis. Notice that our terminology is different from Moldovanu and Sela (2001, 2002). Our *concavity* of output functions corresponds to their *convexity* of cost functions.

Moldovanu and Sela (2002) concluded that, only if the cost function is convex, split-the-prize contests or split-the-contestants contests can be more profitable than the grand winner-take-all contest.

Note that the present paper compares a contest with n participants and k prizes to divided contests. In our setting, under both concave and convex output functions, division can be profitable for the contest designer.

[7]There also exists the following difference. In the all-pay auction, there is no social cost since the total amount of money is unchanged. However, in contests, the total cost of agents' effort and the principal's profit are not the same. Thus, economic contests involve a social cost, as in situations of the war of attrition.

2 The Model

In this section, we describe the ingredients of our model. Consider an economic contest with n risk neutral agents and $k(< n)$ identical prizes. The numbers n and k are exogenously fixed. Each agent i decides his effort e_i to get a prize. Efforts are outlaid simultaneously and independently. All agents' effort levels are observable by the principal and all agents at the end when the agents have already chosen their effort level. There are k winners, each of whom obtains one prize. The common monetary value of each prize, v, is common knowledge.

Each agent has different ability, which affects his marginal cost of effort. The ability of agent i is denoted by θ_i and this is his private information. Here, high θ means high ability and low marginal cost. Abilities are drawn independently from an interval $[0, 1]$ according to the distribution function F that has a continuous and everywhere strictly positive density function f.[8] The distribution function is common knowledge for the principal and all agents. The agent i's cost function is e_i/θ_i.[9] Notice that, in this contest, the only uncertainty for each agent is his opponents' ability levels. The principal does not know any of his agents' abilities *ex ante*.[10]

The payoff of agent i is, $v - e_i/\theta_i$ if he gets a prize, and $-e_i/\theta_i$ if he does not. Each agent chooses his effort in order to maximize his expected payoff, given his own ability, the shape of distribution function F, the number of opponents, the number of prizes, and the value of the prize.

The principal can determine the organizational structure of the contest. If the principal wants to, he can divide this "n participants and k prizes" contest symmetrically into plural contests, when n and k are not mutually prime. Let A be the set of common divisors of n and k, and a representative element of A be denoted by a. The control variable of the principal is the number of division $a \in A$ only, where $a = 1$ implies not-dividing. Here, since the principal does not know his agents' abilities *ex ante*, he randomly assigns his agents to each division. In each division, each agent whose effort level is no lower than the k/a-th place gets a prize. The effect of division is studied in Section 4 and the principal's contest design problem is studied in Section 5.

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[8] We assume that the ability space is $[0, 1]$. This restriction is only for the analytical convenience. We can preserve all our result in any non-negative abilities with bounded support case.

[9] This is the key assumption that permits us to analyze the problem as a variation of all-pay auction. We assume that $e/0 = \infty$ if $e > 0$. This means that the lowest ability agent has an infinite cost for effort. Therefore, he never outlays effort. This setting is similar to Glazer and Hassin (1988)'s one.

[10] Since we pay attention only to the symmetric equilibrium strategies that are strictly increasing with respect to abilities, the principal can find his agents' abilities by using the equilibrium strategies and agents' effort levels *ex post*.

3 Equilibrium Strategies of Agents

In this section, we identify the symmetric Bayesian Nash equilibrium effort strategy in a contest with n participants and k prizes. First of all, we derive the expected payoff of an agent with ability θ_i under the situation where the opponents use the same effort strategy $\beta(\theta)$. Here, we assume that $\beta(\theta)$ is strictly increasing and differentiable.

Lemma 1. *Suppose that all opponents use the same effort strategy $\beta(\theta)$. For the agent i with ability θ_i, his expected payoff under the effort level e is*

$$\pi(e, \beta \mid \theta_i) = v \sum_{t=1}^{k} \binom{n-1}{t-1} (1 - F(g(e)))^{t-1} (F(g(e)))^{n-t} - \frac{e}{\theta_i}, \qquad (1)$$

where $g(\cdot)$ is the inverse function of $\beta(\theta)$.

Proof. See Appendix A (all appendices are available upon request). ▮

Given Lemma 1, we define the following.

Definition 1. *A strategy $\beta(\theta)$ constitutes a symmetric equilibrium if*

$$\forall\theta,\ \beta(\theta) = \arg\max_{e} \pi(e, \beta \mid \theta).$$

In Proposition 1, we characterize an effort strategy that constitutes a symmetric equilibrium.

Proposition 1. *In a symmetric equilibrium, the effort strategy is given by*

$$\begin{aligned} \beta(\theta) &= v(n-k)\binom{n-1}{k-1} \\ &\quad \times \int_0^{\theta} y f(y)(1 - F(y))^{k-1}(F(y))^{n-k-1} dy. \qquad (2) \end{aligned}$$

Proof. See Appendix A. ▮

By using the proof of Proposition 1, we can easily conclude that the symmetric equilibrium is unique.

In the symmetric equilibrium, since $\beta(\theta)$ is strictly increasing, agent i's probability of winning (hereafter $p(\theta_i; n, k)$) is equivalent to the probability that his ability is no lower than the k-th place among n participants, that is,

$$p(\theta_i; n, k) = \sum_{t=1}^{k} \binom{n-1}{t-1} (1 - F(\theta_i))^{t-1} (F(\theta_i))^{n-t}.$$

By using this notation, we can rewrite the effort strategy that constitutes the symmetric equilibrium (i.e., equation (2)) as follows:

$$\beta(\theta) = v \int_0^\theta y \, \frac{\partial p(y; n, k)}{\partial y} dy.$$

Note that, if $v = 1$, this equilibrium strategy is exactly the same as that of the all-pay auction under the following situation: n potential buyers compete for k identical objects; each buyer demands at most one object; and valuations are drawn independently from an interval $[0, 1]$ according to the distribution function F that has a continuous and everywhere strictly positive density function f.

In this section we identify the effort strategy that constitutes the symmetric equilibrium. By using this strategy, in the next section, we study the effects of division of a contest.

4 Division of a Contest

4.1 The Effects on the Agents' Equilibrium Behavior

The aim of this section is to study the effects of division of a contest for the agents' equilibrium behavior. If the principal does not divide the contest with n participants and k prizes (hereafter (n, k)-contest), each agent competes with his $n - 1$ fellow agents for k prizes. If the principal divides the contest into $a(\neq 1)$ contests, each agent competes with his $n/a - 1$ fellow agents for k/a prizes in each assigned $(n/a, k/a)$-contest. Here a is a common divisor of n and k. In the rest of this paper, we use the notation $\beta(\theta; n, k)$ as the effort strategy that constitutes the symmetric equilibrium in a (n, k)-contest. To study the effects of division, we will compare $\beta(\theta; n, k)$ with $\beta(\theta; n/a, k/a)$.

In Proposition 2, the effects of division of a (n, k)-contest is described. Moreover, the comparison of $\beta(\theta; n/a', k/a')$ with $\beta(\theta; n/a, k/a)$ is showed, where $a' > a \geq 1$. This proposition tells us that there exist three and only three patterns of change. In Case 1 and Case 2, division brings about increases in lower ability agents' efforts in exchange for decreases in higher ability agents' efforts. In Case 3, division brings about increases in lower and higher ability agents' efforts in exchange for decreases in middle ability agents' efforts.

Proposition 2. *Division of a contest (or an increase in the number of divisions) changes agents' equilibrium behavior. For $a' > a \geq 1$, there exist three and only three patterns of change as follows:*

Case 1: There exists a threshold $\bar{\theta} \in (0,1)$,

$$\begin{array}{rcll} \beta(\theta; n/a', k/a') & = & \beta(\theta; n/a, k/a) = 0, & \textit{if } \theta = 0, \\ \beta(\theta; n/a', k/a') & > & \beta(\theta; n/a, k/a), & \textit{if } \theta \in (0, \bar{\theta}), \\ \beta(\theta; n/a', k/a') & = & \beta(\theta; n/a, k/a), & \textit{if } \theta = \bar{\theta}, \\ \beta(\theta; n/a', k/a') & < & \beta(\theta; n/a, k/a), & \textit{if } \theta \in (\bar{\theta}, 1]. \end{array}$$

Case 2: There exists a threshold $\bar{\theta} \in (0,1)$,

$$\begin{array}{rcll} \beta(\theta; n/a', k/a') & = & \beta(\theta; n/a, k/a) = 0, & \textit{if } \theta = 0, \\ \beta(\theta; n/a', k/a') & > & \beta(\theta; n/a, k/a), & \textit{if } \theta \in (0, \bar{\theta}), \\ \beta(\theta; n/a', k/a') & = & \beta(\theta; n/a, k/a), & \textit{if } \theta = \bar{\theta}, \\ \beta(\theta; n/a', k/a') & < & \beta(\theta; n/a, k/a), & \textit{if } \theta \in (\bar{\theta}, 1), \\ \beta(\theta; n/a', k/a') & = & \beta(\theta; n/a, k/a), & \textit{if } \theta = 1. \end{array}$$

Case 3: There exist two thresholds that satisfy $0 < \bar{\theta} < \hat{\theta} < 1$,

$$\begin{array}{rcll} \beta(\theta; n/a', k/a') & = & \beta(\theta; n/a, k/a) = 0, & \textit{if } \theta = 0, \\ \beta(\theta; n/a', k/a') & > & \beta(\theta; n/a, k/a), & \textit{if } \theta \in (0, \bar{\theta}), \\ \beta(\theta; n/a', k/a') & = & \beta(\theta; n/a, k/a), & \textit{if } \theta = \bar{\theta}, \\ \beta(\theta; n/a', k/a') & < & \beta(\theta; n/a, k/a), & \textit{if } \theta \in (\bar{\theta}, \hat{\theta}), \\ \beta(\theta; n/a', k/a') & = & \beta(\theta; n/a, k/a), & \textit{if } \theta = \hat{\theta}, \\ \beta(\theta; n/a', k/a') & > & \beta(\theta; n/a, k/a), & \textit{if } \theta \in (\hat{\theta}, 1]. \end{array}$$

Proof. See Appendix A. ▮

We provide examples. Each follows each case of Proposition 2, respectively.

Example 4.1. *Suppose that the environment is* $n = 4$, $k = 2$, $v = 1$.

- *If* $F(\theta) = \theta^2$, *then* $\beta(\theta; 4, 2) = 12(\theta^5/5 - \theta^7/7)$, *and* $\beta(\theta; 2, 1) = 2\theta^3/3$. *In this case, the changes in effort levels brought by division follow Case 1.*

- *If* $F(\theta) = \theta$, *then* $\beta(\theta; 4, 2) = 2\theta^3 - 3\theta^4/2$, *and* $\beta(\theta; 2, 1) = \theta^2/2$. *In this case, the changes in effort levels brought by division follow Case 2.*

- *If* $F(\theta) = 2\theta - \theta^2$, *then* $\beta(\theta; 4, 2) = 6(4\theta^3/3 - 7\theta^4/2 + 18\theta^5/5 - 5\theta^6/3 + 2\theta^7/7)$, *and* $\beta(\theta; 2, 1) = \theta^2 - 2\theta^3/3$. *In this case, the changes in effort levels brought by division follow Case 3.*

These examples are depicted in Figures 1, 2, and 3. Figure 1 depicts the case of $F(\theta) = \theta^2$, Figure 2 depicts the case of $F(\theta) = \theta$, and Figure 3

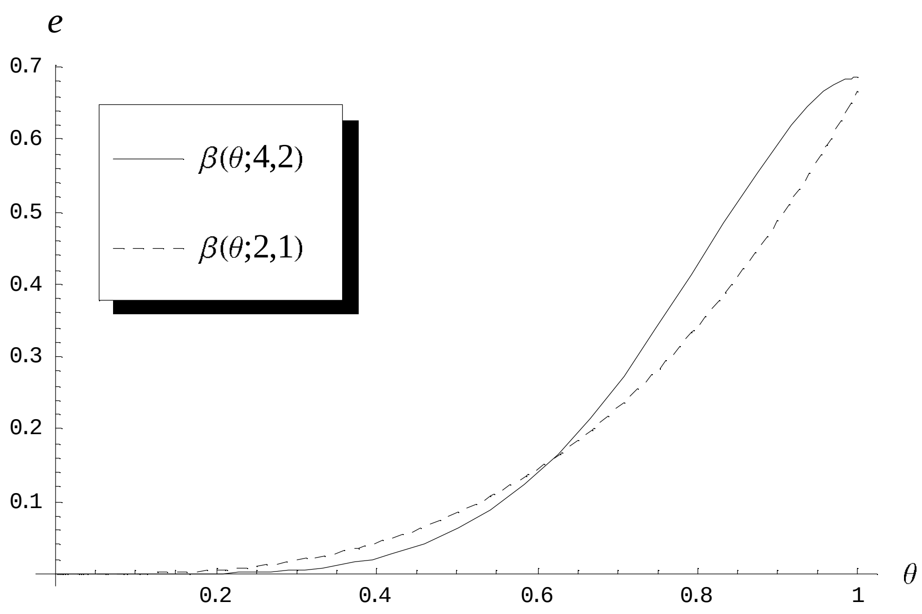

Figure 1:

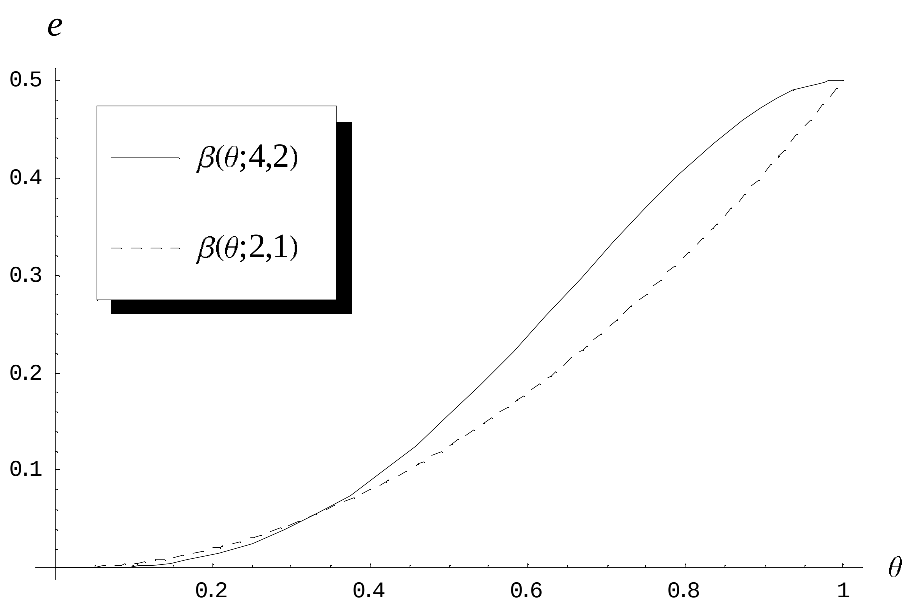

Figure 2:

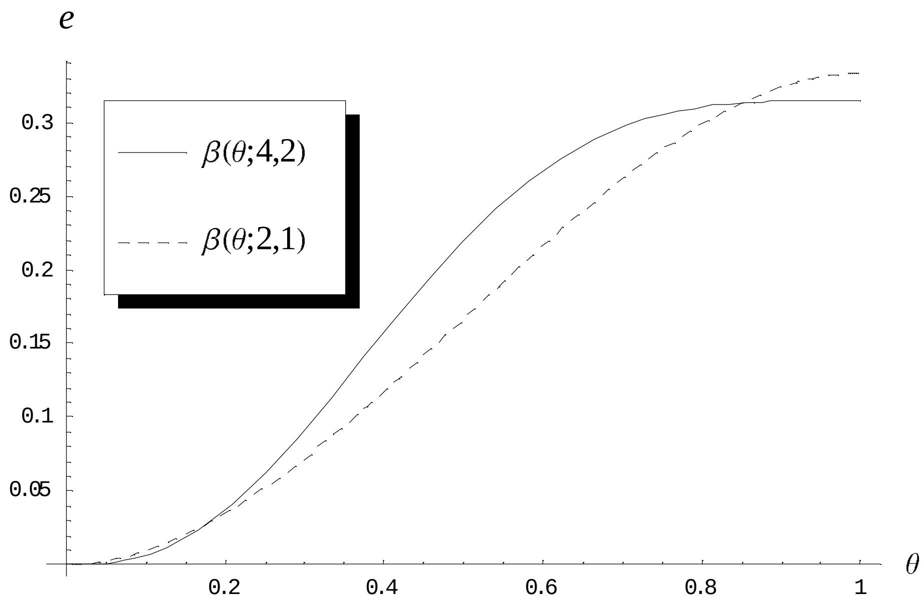

Figure 3:

depicts the case of $F(\theta) = 2\theta - \theta^2$. Horizontal axis is abilities and vertical axis is effort levels.

We briefly describe the reason why division changes agents' equilibrium effort as described in Proposition 2. When the principal divides a contest, winners are not always chosen in the order of the agents' abilities, while winners are chosen in the order of the agents' abilities if the principal does not divide the contest. That is, division makes room for an inefficient allocation of prizes.

An inefficient allocation improves the winning prospect of agents with lower ability.[11] Because the probability of winning of agents at the left tail (note that the probability of winning of an agent with the lowest ability is always zero) increases faster in this situation compared to the situation with efficient allocation, the possibility of inefficient allocation induces more aggressive effort at the left tail.

For middle ability agents, division decreases the gradient of the probability of winning function $p(\theta)$, while division increases the gradient of $p(\theta)$ for lower and higher ability agents. This reflects decreases in the gradient of the equilibrium effort function $\beta(\theta)$ for middle ability agents, and increases in the gradient of $\beta(\theta)$ for higher ability agents.

[11] Another explanation is as follows: Given the ratio of the number of participants to the number of prizes, the winning prospect of lower ability agents is smaller in a larger contest due to the law of large numbers effect.

4.2 The Aggregate Effect of Division

Now we turn our attention to the aggregate effect of division. First of all, we define and consider a regularity condition on the distribution of abilities.

Definition 2. *The distribution function F is regular if*

$$J(\theta) \equiv \theta - \frac{1}{\rho(\theta)}$$

is increasing, where $\rho(\theta) = f(\theta)/(1 - F(\theta))$ is the hazard rate for F.

The regularity of distribution functions is satisfied if the hazard rate increases or does not decrease too rapidly with θ. Many distributions of abilities satisfy this condition, including the distribution functions displayed in Example 4.1.[12]

As mentioned before, the contest game considered in this paper is isomorphic to an all-pay auction. Moreover, this contest game satisfies the conditions of the revenue equivalence theorem: Bidders are risk neutral and have unit-demand, and their types are *i.i.d.* Under the hypotheses of the revenue equivalence theorem and the regularity condition, Maskin and Riley (1989) showed that, in optimal auctions, the winners are chosen in the order of the bidders' valuations (under an appropriate minimum allowable bid rule). As a direct consequence of the above result, we obtain the following result.

Proposition 3. *If the distribution function is regular, division of a contest (or an increase in the number of divisions) decreases the expected total effort of the agents.*

Proof. See the proof of Proposition 4 in Maskin and Riley (1989). ▮

Proposition 3 implies that division decreases the expected effort per agent. This is the key finding that is used in Section 6.

To sum up, division changes agents' behavior as described in Proposition 2, while it decreases the expected effort per agent (under the regularity condition). Hence, in the case where the principal welcomes these consequences, he divides the contest.

5 The Principal's Contest Design

In this section, we demonstrate that division is sometimes profitable for the contest designer. We assume that the profit function of the principal is

$$\pi(e_1, \ldots, e_n) = \sum_i \gamma(e_i),$$

[12] See Remark 8.1 in Maskin and Riley (1984) for a discussion on the regularity condition.

where $\gamma(\cdot)$ is the output function that transforms each agent's effort to his output. We also assume that the function $\gamma(\cdot)$ is strictly increasing. We consider the cases where $\gamma(\cdot)$ is a linear, concave, or convex function.[13]

The optimal contest structure is characterized by $a^* \in A$, where

$$a^* \in \arg\max_{a \in A} \int_0^1 \gamma(\beta(\theta; n/a, k/a)) f(\theta)\, d\theta.$$

Here $\int_0^1 \gamma(\beta(\theta; n/a, k/a)) f(\theta)\, d\theta$ is the expected output per agent in the $(n/a, k/a)$-contest.

Firstly we describe the case where $\gamma(\cdot)$ is linear. By using Proposition 3, we obtain the following.

Proposition 4. *Under the regularity condition, if $\gamma(\cdot)$ is linear, division is never profitable for the contest designer.*

Proof. When $\gamma(\cdot)$ is linear, the maximization of the sum of agents' output is equivalent to the maximization of the sum of agents' effort. From Proposition 3, division of a contest decreases the expected sum of agents' effort, and therefore it is never profitable for the contest designer. ▮

In this linear case, the maximization of total output is equivalent to the maximization of total effort, so that we can apply the results of optimal auction design to the study of optimal contest design and an efficient allocation of prizes (i.e., not-dividing) is optimal.

Secondly we describe the case where $\gamma(\cdot)$ is concave.

Proposition 5. *When $\gamma(\cdot)$ is concave, division of a contest can be profitable for the contest designer, even if the regularity condition holds.*

Proof. See Appendix A. ▮

An intuitive explanation of Proposition 5 is as follows: If the output function $\gamma(\cdot)$ is a concave function of effort, the principal prefers to have a low effort agent increase one unit of his effort in exchange for a high effort agent reducing one unit of his effort. Therefore contests that increase the effort of low effort types and reduce the effort of high effort types are more likely to be chosen under a larger degree of the concavity of the output function.

In this non-linear case, the maximization of total output is no longer equivalent to the maximization of total effort. Therefore, the results of optimal auction design are not applicable to the study of optimal contest design. That is to say, an inefficient allocation of prizes can be profitable, even if the regularity condition holds.

The following example illustrates the situation with a concave output function.

[13]Other objectives of the principal are also convincing depending on the situation. We will discuss the other objectives of the contest designer later.

Example 5.1. *We consider a* $(8,4)$*-contest with* $v = 1$*. Agents' abilities are uniformly distributed. The output function is*

$$\gamma(e_i) = -\exp\{-\alpha e_i\} + 1,$$

where $\alpha > 0$*. The Arrow-Pratt measure* $(m(e_i) = -\gamma''(e_i)/\gamma'(e_i))$ *of this output function is equal to the constant* $\alpha(> 0)$ *at all* e_i*. Here, a larger* $m(= \alpha)$ *implies that the degree of concavity is larger.*[14] *In this circumstance,*

- *If* $\alpha = 1$, $a^* = 1$ *(i.e., division is not profitable).*
- *If* $\alpha = 10$, $a^* = 2$.
- *If* $\alpha = 20$, $a^* = 4$.

Thus, if $\gamma(\cdot)$ *is concave enough, division can be profitable.*

This example tells us that, given the value of each prize v, a larger α is required for division to be profitable. Note that, because there is linear relation between v and e_i (Proposition 1), we can also conclude that, for a given α, a larger v is required for division to be profitable. Figures 4 depicts above example. Horizontal axis is α, and vertical axis is the expected output per agent, which is denoted by x.

Lastly we describe the case where $\gamma(\cdot)$ is convex.

Proposition 6. *Suppose that* $\gamma(\cdot)$ *is convex and the regularity condition holds.*

- *If changes in agents' equilibrium behavior by division follow Case 1 or Case 2 of Proposition 2, division is never profitable.*
- *The fact that changes in agents' equilibrium behavior by division follow Case 3 of Proposition 2 is a necessary condition for division to be profitable.*

Proof. We can prove this proposition with a similar method used in the proof of Proposition 5, thus this proof is omitted. ∎

An intuitive explanation of Proposition 6 is as follows: If the output function $\gamma(\cdot)$ is a convex function of effort, the principal prefers to have a high effort agent increase one unit of his effort in exchange for a low effort agent reducing one unit of his effort. In Case 1 and Case 2 of Proposition 2, because division decreases the effort of high effort types and increases the effort of low effort types, it is never profitable. Only in Case 3 of Proposition 2, division increases the effort of high effort types. Therefore the fact that changes

[14]The Arrow-Pratt measure of curvature is usually used to compare risk attitude of agents.

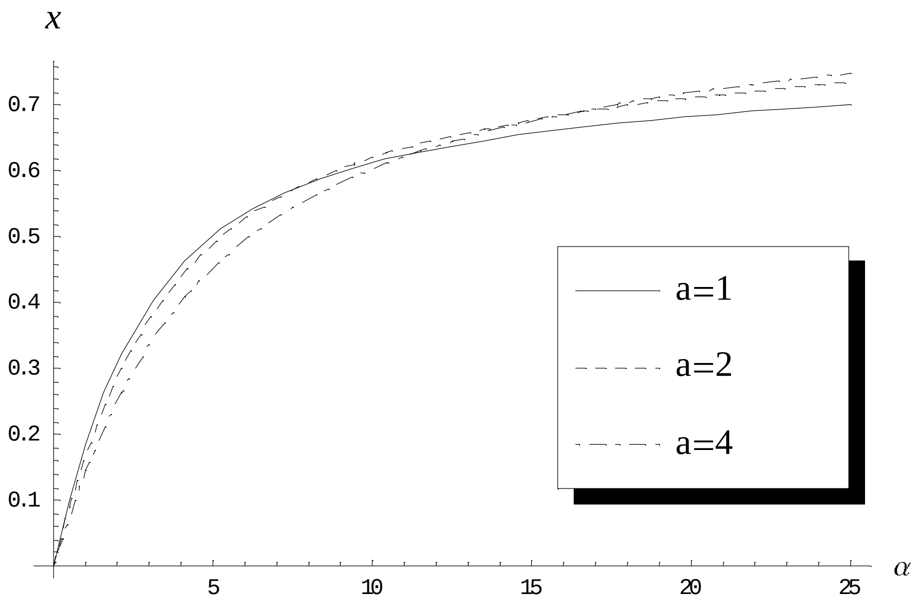

Figure 4:

in agents' equilibrium behavior by division follow Case 3 of Proposition 2 is a necessary condition for division to be profitable.

By using the fact that changes in agents' effort that are brought by division always obey Case 2 of Proposition 2 if abilities are uniformly distributed,[15] we obtain the following as a corollary of Proposition 6.

Corollary 1. *If abilities are uniformly distributed and $\gamma(\cdot)$ is convex, division is never profitable for the contest designer.*

The following example illustrates the situation where division is profitable for the contest designer under a convex output function.

Example 5.2. *We consider a $(4, 2)$-contest with $v = 100$ under $F(\theta) = 2\theta - \theta^2$. The output function is*

$$\gamma(e_i) = \exp\{e_i\} - 1.$$

In this circumstance, $a^* = 2$. Thus, there exist circumstances that division is profitable for the contest designer under a convex output function.

In this section, we consider the optimal contest structure under a specific objective of the contest designer and demonstrate that division is sometimes profitable for the contest designer.

[15]The proof of this statement is provided in Appendix B.

As mentioned before, there exist many possible objectives of the principal in contests. For example, the principal may desire to maximize the minimum effort among the participants in contests. This objective is suitable if the production function of the principal is a Leontief production function, and may be suitable if the contest designer attaches greater importance to the closeness of contestants efforts as well as a high average effort (as in a sport contest).[16] Under this objective function, division is sometimes profitable, because it increases the effort of low effort types. For example, consider a (4,2)-contest with a Leontief production function. If $F(\theta) = \theta^2$, division is profitable for the principal, while division is not profitable if $F(\theta) = \theta$ or $F(\theta) = 2\theta - \theta^2$.

We can describe another situation where the contest designer is willing to divide a contest. Suppose that, for each agent, if his effort exceeds a given minimum requirement level $\underline{e}(> 0)$, a positive constant output is realized. In this environment, the principal designs the contest structure to maximize the expected number of participants whose efforts exceed the minimum requirement level. Since division of a contest increases efforts of lower effort types, we can conclude that division is profitable for the contest designer if the minimum requirement level is sufficiently small.

In the next section, we apply the result to the class size puzzle.

6 An Application to Educational Attainment

The relation of educational attainment to class size is studied in a large literature. It has been expected that class size reductions yield an increase in educational productivity. For example, since class size reductions yield relaxation of the congestion effects in classrooms, education may become efficient. This positive effect is called the class size effect. However, many empirical studies[17] conclude that no significant class size effect exists. This inconsistency is called the class size puzzle.

Lazear (2001) presented one of the reasons why it is difficult to find class size effect in empirical data. He presented a disruption model of educational production and paid attention to the case where the class size is a choice variable of each school. He concluded that there exists a positive correlation between students' abilities and the optimal class size and therefore it is difficult to find class size effect in empirical data.

Our discussion exhibits another explanation of the class size puzzle. Consider the following situation. There exists a school with n students in each grade. Each student has different ability, which affects his additional cost of study. We assume that students study to obtain the higher grade. Here

[16] Singh and Wittman (1998) studied two player contests and compared the contest designer's optimal reward for winning under different objectives.

[17] One of the recent studies is Hoxby (2000).

other motivations to study are ignored. The monetary value of obtaining each grade is the same among all students. For the present, we assume that the values are exogenously given. If the same grade is obtained, less effort is profitable for students.

Students are graded on a curve and grading is conducted in each classroom.[18][19] To simplify, we assume that there exist two distinct grades, Pass and Fail. Only k students will pass each year. The model of our contest game is suitable for the above situation.[20] Here, we can regard a reduction of class size as *an increase in the number of divisions*.

Now, we apply our results to the above situation. Proposition 3 implies that a reduction of class size decreases the expected effort per student (under the regularity condition). Note that, if a fixed proportion of students pass in each class, an increase in the number of classes decreases the degree of selection efficiency. Clearly, if all students belonged to one big class, the k most able students pass. If the students are divided into several classes, and if the proportion of passes to fails in each class is fixed, an inefficient selection occurs. The reason is obvious, as too many students pass from poor classes and too few from good classes.

Additionally, a reduction of class size may decrease the monetary value of obtaining the higher grade. Grades are presumably valuable to the student in so far as they are effective signal of ability or effort. The possibility of inefficient selection weakens the value of the higher grade as a good signal, and then it may decrease the monetary value of obtaining the higher grade. Obviously, a reduction of the value of prize involves overall reduction of efforts.

As discussed above, class size reductions may decrease educational productivity if grading is conducted in this manner.[21] Hence, we suggest that, to solve the class size puzzle, we have to consider not just the relation between educational attainment and class size but the relation among educational attainment, class size, and the method of grading. Lazear (2001) said "Japan has high test score and large class size."[22] This fact is consistent to

[18] Until the school year 2001, grading in Japanese public elementary schools had been conducted in this manner. From the school year 2002, grading is conducted with absolute scales.

[19] In Japanese public elementary schools, each student belongs to assigned classroom and most subjects are taught in the classroom as a unit. Moreover, in each classroom, these subjects are taught by one teacher in charge. Each class teacher grades his students with a given method.

[20] Virtually, there exists no literature that studies educational productivity by using principal-agent models. Exceptions include Becker and Rosen (1992) and Betts (1998). Becker and Rosen (1992) compared grading on a curve with grading on absolute scales. Betts (1998) studied the relation of educational standards to inequality.

[21] Obviously, these effects do not appear if grading is conducted with absolute scales.

[22] See footnote 15 of Lazear (2001). See also Wößmann and West (2002). They concluded as follows: "While we find sizable beneficial effects of smaller classes in Greece and Iceland, the possibility of even small effects is rejected in Japan and Singapore."

our findings.

7 Conclusion

In this paper, we described the effects of division of a contest with identical prizes. The fundamental reason why division changes the equilibrium behavior of agents is changes in each type's probability of winning due to the possibility of inefficient allocation of prizes. Additionally, we showed that division is sometimes profitable for the contest designer, in the sense that division brings about increases in lower ability agents' (and occasionally higher ability agents') efforts in exchange for decreases in other agents' efforts.

This paper considers specific objectives of the principal. The purpose of the principal is assumed to be providing appropriate incentives to agents. Naturally, we do not exclude other reasons why the contest designer divides a contest. Let us consider a competition for promotion in a company. If whether or not the promoted employee accumulated professional knowledge is important, the competition for promotion may be conducted in each department. Another reason is that, if it is difficult to compare employees who were assigned to different jobs, the competition for promotion may be conducted among employees who took the same job. However, irrespective of the principal's intention of dividing, division of a contest changes the participants' behavior as described in this paper.

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8 Appendix A (Not for Publication)

Proof of Lemma 1. Given that the agent i's ability is θ, his effort level is e, and all other agents $j \neq i$ outlay effort levels $\beta(\theta_j)$, agent i's expected payoff is

$$\begin{aligned}
\pi(e, \beta \mid \theta) \quad = \quad & v \times (\ \text{Prob}(e > \beta(\theta_j), \forall j \neq i) \\
& + \text{Prob}(\text{largest effort among all } j \neq i > e \\
& \qquad \geq \text{second largest effort among all } j \neq i) \\
& \qquad \vdots \\
& + \text{Prob}(\ k-1\text{th largest effort among all } j \neq i > e \\
& \qquad \geq k\text{th largest effort among all } j \neq i)\ ) \\
& - \frac{e}{\theta}.
\end{aligned}$$

Since we assume that $\beta(\cdot)$ is strictly increasing and differentiable, we can define $g(\cdot)$ as the inverse function of $\beta(\cdot)$. Then we can rewrite the above equation as follows:

$$\begin{aligned}
\pi(e, \beta \mid \theta) \quad = \quad & v \times (\ \text{Prob}(g(e) > \theta_j, \forall j \neq i) \\
& + \text{Prob}(\text{largest type among all } j \neq i > g(e) \\
& \qquad \geq \text{second largest type among all } j \neq i) \\
& \qquad \vdots \\
& + \text{Prob}(\ k-1\text{th largest type among all } j \neq i > g(e) \\
& \qquad \geq k\text{th largest type among all } j \neq i)\ ) \\
& - \frac{e}{\theta}.
\end{aligned}$$

Since the distribution function F is *i.i.d.*, we obtain

$$\begin{aligned}
\pi(e, \beta \mid \theta) \quad = \quad & v \times (\ (F(g(e)))^{n-1} \\
& + (n-1)(1 - F(g(e)))(F(g(e)))^{n-2} \\
& \qquad \vdots \\
& + \binom{n-1}{k-1} (1 - F(g(e)))^{k-1} (F(g(e)))^{n-k}\ ) \\
& - \frac{e}{\theta} \\
= \quad & v \times \left(\sum_{t=1}^{k} \binom{n-1}{t-1} (1 - F(g(e)))^{t-1} (F(g(e)))^{n-t} \right) - \frac{e}{\theta}.
\end{aligned}$$

Thus expression (1) was derived. ∎

Proof of Proposition 1. To simplify the notation, we define

$$\Psi_t(x) = \binom{n-1}{t-1}(1-F(x))^{t-1}(F(x))^{n-t}.$$

$\Psi_t(x)$ is the probability that x is the t-th largest value among n variables.

The first order condition of the maximization problem of agent i (i.e., F.O.C. of the maximization of expression (1) with respect to e) is

$$v\sum_{t=1}^{k}\Psi_t'(g(e))g'(e) - \frac{1}{\theta} = 0. \tag{3}$$

In the symmetric equilibrium, since $g(e) = \theta$ and $g'(e) = (\beta'(\theta))^{-1}$, equation (3) is equivalent to

$$v\sum_{t=1}^{k}\Psi_t'(\theta)\frac{1}{\beta'(\theta)} - \frac{1}{\theta} = 0 \;\Leftrightarrow\; \beta'(\theta) = \theta\; v\sum_{t=1}^{k}\Psi_t'(\theta).$$

We can rewrite the above as follows:

$$\beta'(\theta) = \theta\; v(n-k)\binom{n-1}{k-1}f(\theta)(1-F(\theta))^{k-1}(F(\theta))^{n-k-1}. \tag{4}$$

Integration with the boundary condition ($\beta(0) = 0$) yields

$$\begin{aligned}\beta(\theta) \;&=\; v(n-k)\binom{n-1}{k-1}\\ &\quad\times\int_0^{\theta} yf(y)(1-F(y))^{k-1}(F(y))^{n-k-1}dy. \end{aligned}\tag{5}$$

We can easily check the strict increase and differentiability of $\beta(\theta)$ by equation (4).

We will show that for any type of agents no one can gain by deviating from the strategy $\beta(\theta)$. Since the first order condition is satisfied in the equilibrium, for the agent whose ability is θ', $\partial\pi(\theta', e')/\partial e = 0$ is satisfied. Here $e' = \beta(\theta')$. We have to check the following conditions.

$$\frac{\partial\pi}{\partial e}(\theta', e)\begin{cases} > 0 & \text{if } e < \beta(\theta') = e' \\ < 0 & \text{if } e > \beta(\theta') = e'. \end{cases}$$

Since the function $\pi(\theta, e)$ is continuous in e, the above expression implies that $\pi(\theta, e)$ is maximized at $e = \beta(\theta)$. First we verify $\partial\pi(\theta', e)/\partial e > 0$ if $e < \beta(\theta') = e'$. Take $\hat{e}$ as an effort smaller than the equilibrium effort level e' for the agent whose ability is θ'. Here we have to check $\partial\pi(\theta', \hat{e})/\partial e > 0$. The ability type which takes $\hat{e}$ in the equilibrium is denoted by $\hat{\theta}$. This means that $\beta(\hat{\theta}) = \hat{e}$. Since $\beta(\theta)$ is strictly increasing and $\hat{e} < e'$, we obtain $\hat{\theta} < \theta'$. Because

$$\frac{\partial \pi}{\partial e}(\theta, e) = v \sum_{t=1}^{k} \Psi'_t(g(e))g'(e) - \frac{1}{\theta},$$

$\partial^2 \pi(\theta, e)/\partial\theta\partial e = 1/\theta^2 > 0$. Therefore, for any e, $\partial\pi(\theta', e)/\partial e > \partial\pi(\hat{\theta}, e)/\partial e$ is shown. Since the above is satisfied at $\hat{e}$, $\partial\pi(\theta', \hat{e})/\partial e > \partial\pi(\hat{\theta}, \hat{e})/\partial e$. Since the right hand side is zero by the first order condition, $\partial\pi(\theta', \hat{e})/\partial e > 0$ is shown. With a similar argument, the other side can be shown too. Therefore expression (5) is the equilibrium strategy in a symmetric equilibrium. ▮

Proof of Proposition 2. In this proof, for the expositional convenience, we compare a $(2n, 2k)$-contest with a (n, k)-contest. The general case can be proven in the same way.[23]

From Proposition 1,

$$\begin{aligned} &\beta(\theta; 2n, 2k) - \beta(\theta; n, k) \\ &= v(2n-2k)\binom{2n-1}{2k-1}\int_0^\theta yf(y)(1-F(y))^{2k-1}(F(y))^{2n-2k-1}dy \\ &\quad -v(n-k)\binom{n-1}{k-1}\int_0^\theta yf(y)(1-F(y))^{k-1}(F(y))^{n-k-1}dy \qquad (6) \\ &= v(n-k)\int_0^\theta yf(y)(1-F(y))^{k-1}(F(y))^{n-k-1} \\ &\quad \times \left(2\binom{2n-1}{2k-1}(1-F(y))^k(F(y))^{n-k} - \binom{n-1}{k-1}\right)dy. \qquad (7) \end{aligned}$$

Since $v(n-k)$ is a strictly positive constant, we have to pay attention only to the sign of the rest of above expression. To see the shape of the integrand of expression (7) in the interval $[0, 1]$, we divide this function into three parts,

$$\begin{aligned} &y,\ f(y)(1-F(y))^{k-1}(F(y))^{n-k-1}, \\ &\quad \text{and } \left(2\binom{2n-1}{2k-1}(1-F(y))^k(F(y))^{n-k} - \binom{n-1}{k-1}\right), \end{aligned}$$

and study their characteristics. First, y is a strictly increasing function and non-negative in the interval $[0, 1]$. Second, since the density function f is continuous and everywhere strictly positive, $f(y)(1-F(y))^{k-1}(F(y))^{n-k-1} > 0$ in the interval $(0, 1)$. Third,

$$2\binom{2n-1}{2k-1}(1-F(y))^k(F(y))^{n-k} - \binom{n-1}{k-1}$$

[23] The comparison of a $(n/a', k/a')$-contest with a $(n/a, k/a)$-contest is equivalent to the comparison of a (n, k)-contest with a (tn, tk)-contest, where $t = a'/a > 1$. The general case can be proven by replacing "2" of the present proof with "t".

is strictly negative in the interval $[0, y')$, strictly positive in the interval (y', y''), and strictly negative in the interval $(y'', 1]$, where $y' \in (0, 1)$ and $y'' \in (y', 1)$ are certain thresholds. Here, $\int_0^1 \Phi(y)dy = 0$, which is stated later, implies the existence of interval (y', y'').

From these characteristics, the shape of the integrand in the interval $[0, 1]$ is (1) zero at $y = 0$, (2) strictly negative in the interval $(0, y')$, (3) zero at $y = y'$, (4) strictly positive in the interval (y', y''), (5) zero at $y = y''$, and strictly negative in the interval $(y'', 1)$. Here, for the convenience of notation, we define

$$\Phi(y) = f(y)(1 - F(y))^{k-1}(F(y))^{n-k-1} \times \left(2\binom{2n-1}{2k-1}(1 - F(y))^k(F(y))^{n-k} - \binom{n-1}{k-1}\right).$$

Because $\int_0^1 \Phi(y)dy = 0$,[24] there exists $\tilde{y} \in (y', y'')$, $\int_0^{y'} \Phi(y)dy + \int_{y'}^{\tilde{y}} \Phi(y)dy = 0$ and $\int_{\tilde{y}}^{y''} \Phi(y)dy + \int_{y''}^1 \Phi(y)dy = 0$. This is equivalent to $\int_0^{\tilde{y}} \Phi(y)dy = \int_{\tilde{y}}^1 \Phi(y)dy = 0$.

The integrand of expression (7) can be written as $y\Phi(y)$. Since y is strictly increasing and non-negative in the interval $[0, 1]$, there exist three possible cases as follows:

Case 1: $\exists \bar{y} \in (y', y'')$,
$\int_0^{y'} y\Phi(y)dy + \int_{y'}^{\bar{y}} y\Phi(y)dy = 0$ and $\int_{\bar{y}}^{y''} y\Phi(y)dy + \int_{y''}^1 y\Phi(y)dy > 0$,

Case 2: $\exists \bar{y} \in (y', y'')$,
$\int_0^{y'} y\Phi(y)dy + \int_{y'}^{\bar{y}} y\Phi(y)dy = 0$ and $\int_{\bar{y}}^{y''} y\Phi(y)dy + \int_{y''}^1 y\Phi(y)dy = 0$,

Case 3: $\exists \bar{y} \in (y', y'')$,
$\int_0^{y'} y\Phi(y)dy + \int_{y'}^{\bar{y}} y\Phi(y)dy = 0$ and $\int_{\bar{y}}^{y''} y\Phi(y)dy + \int_{y''}^1 y\Phi(y)dy < 0$.

In Case 1, for all $\theta \in (0, \bar{y})$, $\beta(\theta; 2n, 2k) - \beta(\theta; n, k) < 0$, and for all $\theta \in (\bar{y}, 1]$, $\beta(\theta; 2n, 2k) - \beta(\theta; n, k) > 0$. In Case 2, for all $\theta \in (0, \bar{y})$, $\beta(\theta; 2n, 2k) - \beta(\theta; n, k) < 0$, and for all $\theta \in (\bar{y}, 1)$, $\beta(\theta; 2n, 2k) - \beta(\theta; n, k) > 0$. In Case 3, there exists $\hat{y} \in (\bar{y}, 1)$, $\int_0^{\bar{y}} y\Phi(y)dy = \int_{\bar{y}}^{\hat{y}} y\Phi(y)dy = 0$ and $\int_{\hat{y}}^1 y\Phi(y)dy < 0$. This is equivalent to, for all $\theta \in (0, \bar{y})$, $\beta(\theta; 2n, 2k) - \beta(\theta; n, k) < 0$, for all $\theta \in (\bar{y}, \hat{y})$, $\beta(\theta; 2n, 2k) - \beta(\theta; n, k) > 0$, and for all $\theta \in (\hat{y}, 1]$, $\beta(\theta; 2n, 2k) - \beta(\theta; n, k) < 0$. Thus we conclude that only these three cases are possible.

What is left to prove is to verify the existence of these three cases. The existence can be proven by Example 4.1. Hence, we obtain Proposition 2. ▮

[24]Since expression (6) is equivalent to $v\int_0^\theta y\frac{\partial p(y;2n,2k)}{\partial y}dy - v\int_0^\theta y\frac{\partial p(y;n,k)}{\partial y}dy$, $\int_0^1 \Phi(y)dy = \left(\int_0^1 \frac{\partial p(y;2n,2k)}{\partial y}dy - \int_0^1 \frac{\partial p(y;n,k)}{\partial y}dy\right)/(v(n-k)) = (1-1)/(v(n-k)) = 0$.

Proof of Proposition 5. Let $\hat{a} \in A$ be the number of divisions where $\hat{a} > 1$. From Proposition 2, there exists $\hat{\theta} \in (0,1)$ such that $\beta(\theta; n/\hat{a}, k/\hat{a}) > \beta(\theta; n, k)$ for all $\theta \in (0, \hat{\theta})$. Moreover, there exists $\bar{\theta} \in (\hat{\theta}, 1]$ such that $\beta(\theta; n/\hat{a}, k/\hat{a}) < \beta(\theta; n, k)$ for all $\theta \in (\hat{\theta}, \bar{\theta})$.

Proposition 3 implies

$$\int_0^{\hat{\theta}} (\beta(\theta; n/\hat{a}, k/\hat{a}) - \beta(\theta; n, k)) f(\theta) d\theta$$
$$< \int_{\hat{\theta}}^{\bar{\theta}} (\beta(\theta; n, k) - \beta(\theta; n/\hat{a}, k/\hat{a})) f(\theta) d\theta. \tag{8}$$

Let e_a and e_b be distinct effort levels that satisfy $e_b > e_a > 0$. If $\gamma(\cdot)$ is strictly concave, $\gamma(e_a + x) - \gamma(e_a) > \gamma(e_b + x) - \gamma(e_b)$, where x is a positive constant. This implies that a lower ability agent's additional effort is more effective for the principal's profit than a higher agent's, since the symmetric equilibrium strategy is strictly increasing. Hence, by division, the left hand side of inequality (8) is relatively amplified. Therefore, we can conclude that, if the degree of concavity of $\gamma(\cdot)$ is large enough, division becomes profitable. ▮

9 Appendix B (Not for Publication)

In Proposition 7, we describe the effort strategy that constitutes the symmetric equilibrium under the uniformly distributed abilities. This proposition implies that, if abilities are uniformly distributed, the changes in equilibrium behavior that are brought by division follow Case 2 of Proposition 2, because the equilibrium effort level of an agent with the highest possible ability is always $v(1 - k/n)$.

Proposition 7. *If abilities are uniformly distributed,*

$$\beta(\theta; n, k) = v\left(1 - \frac{k}{n}\right) \sum_{t=0}^{k-1} \binom{n}{t} (1-\theta)^t \theta^{n-t}. \tag{9}$$

Proof. From Proposition 1 and $F(\theta) = \theta$,

$$\beta(\theta; n, k) = v(n-k) \binom{n-1}{k-1} \int_0^{\theta} y(1-y)^{k-1} y^{n-k-1} dy.$$

Using integration by parts, we obtain

$$\begin{aligned}
\beta(\theta; n, k) &= v(n-k)\binom{n-1}{k-1} \\
&\quad \times \left[\left[-\frac{1}{k}(1-y)^k y^{n-k}\right]_0^\theta - \int_0^\theta \left(-\frac{1}{k}(1-y)^k\right)(n-k)y^{n-k-1}dy\right] \\
&= -\frac{v(n-k)}{k}\binom{n-1}{k-1}(1-\theta)^k\theta^{n-k} \\
&\quad + v(n-k)\binom{n-1}{k-1}\frac{(n-k)}{k}\int_0^\theta (1-y)^k y^{n-k-1}dy. \qquad (10)
\end{aligned}$$

Since

$$\frac{(n-1)!}{(k-1)!(n-k)!} \times \frac{n-k}{k} = \frac{(n-1)!}{k!(n-k-1)!} = \binom{n-1}{k},$$

we obtain

$$\begin{aligned}
&v(n-k)\binom{n-1}{k-1}\frac{(n-k)}{k}\int_0^\theta (1-y)^k y^{n-k-1}dy \\
&= v(n-k)\binom{n-1}{k}\int_0^\theta (1-y)^k y^{n-k-1}dy. \qquad (11)
\end{aligned}$$

By using equation (11) and

$$\beta(\theta; n, k+1) = v(n-k-1)\binom{n-1}{k}\int_0^\theta (1-y)^k y^{n-k-1}dy,$$

we can rewrite equation (10) as follows:

$$\begin{aligned}
\beta(\theta; n, k) &= -\frac{v(n-k)}{k}\binom{n-1}{k-1}(1-\theta)^k\theta^{n-k} \\
&\quad + \frac{n-k}{n-k-1}\beta(\theta; n, k+1).
\end{aligned}$$

The above equation is equivalent to

$$\frac{\beta(\theta; n, k+1)}{n-(k+1)} - \frac{\beta(\theta; n, k)}{n-k} = \frac{v}{k}\binom{n-1}{k-1}(1-\theta)^k\theta^{n-k}.$$

Since

$$\frac{1}{k}\binom{n-1}{k-1} = \frac{1}{n}\binom{n}{k},$$

we obtain

$$\begin{aligned}\sum_{t=1}^{k-1}\left(\frac{\beta(\theta;n,t+1)}{n-(t+1)}-\frac{\beta(\theta;n,t)}{n-t}\right) &= \frac{\beta(\theta;n,k)}{n-k}-\frac{\beta(\theta;n,1)}{n-1} \\ &= \sum_{t=1}^{k-1}\frac{v}{n}\binom{n}{t}(1-\theta)^t\theta^{n-t}. \qquad (12)\end{aligned}$$

Due to the fact that $\beta(\theta;n,1)=v(n-1)\theta^n/n$, equation (12) is equivalent to

$$\frac{\beta(\theta;n,k)}{n-k}-\frac{v\theta^n}{n}=\sum_{t=1}^{k-1}\frac{v}{n}\binom{n}{t}(1-\theta)^t\theta^{n-t}.$$

Therefore, we obtain

$$\beta(\theta;n,k)=v\sum_{t=0}^{k-1}\frac{n-k}{n}\binom{n}{t}(1-\theta)^t\theta^{n-t}.$$

Thus expression (9) was derived. ▮